Computing multi-pulse interactions in the Complex Ginzburg Landau equation

Jamie Chavez Malacara

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The 1D quintic Complex Ginzburg Landau Equation (CGL) is a nonlinear PDE given by $u_t = \alpha u_{xx} + \beta u + \gamma u|u|^2 + \delta u|u|^4$ where $\alpha = \alpha_R + i\alpha_I, \beta = \beta_R + i\beta_I, \gamma = \gamma_R + i\gamma_I, \delta = \delta_R + i\delta_I, \alpha_R > 0$ and $\beta_R < 0$ which is used to study pulses. When studying multi-pulse solutions a difficulty arises - the pulses generally interact with each other, but the interaction usually takes place by the pulse's tails (which are usually exponentially decaying) and is exponentially small with respect to the distance between pulses. This means that direct numerical methods are usually very inefficient for computing the resulting interaction and delicate dynamical properties. The aim of the project was therefore to implement a more efficient numerical method using chebfun, an open source matlab package that uses piecewise polynomial interpolation to perform numerical computations with functions to about 15-digit accuracy.

The first step in the project was to use chebfun to model single pulse solutions.

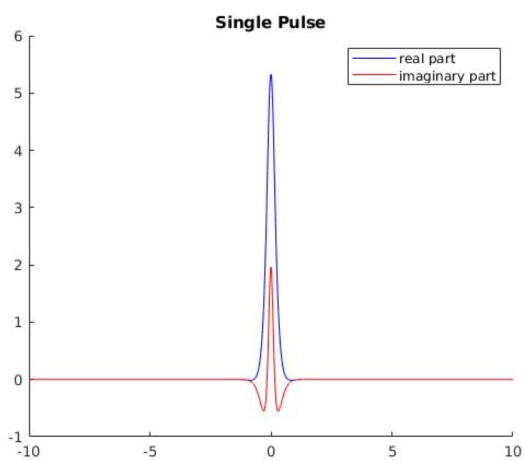

Figure 1: Parameters used: aR = 1/2, aI = 1/2, bR = -0.05, gR = 1.8, gI = 1, dR = -0.05, dI = 0.05, domain = [-10,10]

Once this was done, the next step was implementing the numerical method developed by Tasos Rossides and David Lloyd.

We first linearize the problem. The linear operator is given by $\mathcal{L} := \alpha\partial_x^2 + \beta + f'(V)$, $\mathcal{L}w = \alpha\partial_x^2 w + \beta w + \theta(|V|^2)w + |V|^2\theta'(|V|^2)w + V^2\theta'(|V|^2)\bar{w}$ where $\theta(z) = \gamma z + \delta z^2$ and $f := \gamma V|V|^2 + \delta V|V|^4$. The eigenfunctions of $\mathcal{L}$ corresponding to the zero eigenvalues are $\varphi^r(x) := \partial_r V_\xi(x)|_{r=0} = -V'(x), \varphi^g(x) := \partial_g V_\xi(x)|_{g=0} = iV(x)$ with $V_\xi = e^{ig}V(x-r)$ and r,g constants.

We define the inner product as $< v, w >:= \mathcal{R}(\int_\infty^\infty v(x)\bar{w}(x)dx)$ so the adjoint operator is given by $\mathcal{L}^* = \bar{\alpha}\partial_x^2 + \bar{\beta} + [f'(V)]^*$ and the adjoint eigenfunctions $\psi^r(x).\psi^g(x)$ can be normalized such that $< \phi^r, \psi^r >=< \phi^g, \psi^g >= 1, < \phi^r, \psi^g >=< \phi^g, \psi^r >= 0$. ϕ^r and ψ^r are both odd and ϕ^g and ψ^g are even. We then define the shifted versions associated with V_ξ:

$$\varphi_\xi^r(x) := e^{ig}\varphi^r(x-r), \varphi_\xi^g(x)$$
$$\psi_\xi^r := e^{ig}\psi^g(x-r)e^{ig}\psi^r(x-r), \psi_\xi^g(x) := e^{ig}\psi^g(x-r)$$
$$(P_\xi w)(x) :=< w, \psi\xi^r > \varphi_\xi^r + < w, \psi\xi^g > \varphi_\xi^g$$

where the shifted functions also satisfy the same normalization.

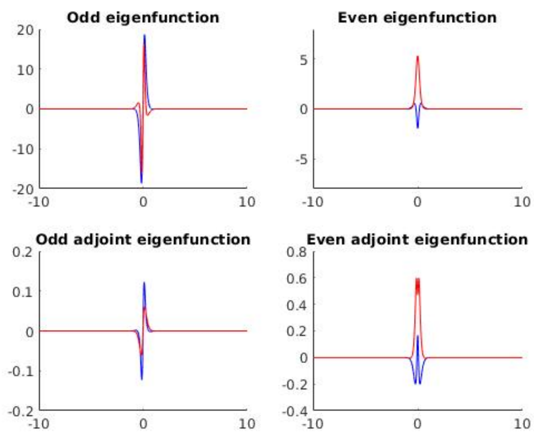

Figure 2: Real parts plotted in blue, imaginary parts plotted in red. All eigenfunctions normalized as described above

Define $V_{\vec{\xi}} := V_{\xi_1}(x) + V_{\xi_2}(x)$ where $\xi_i = (r_i, g_i)$ and $\bar{r}, \bar{g}$ come from

$$\begin{cases} \frac{d}{dt}\bar{r} = Je^{-c\bar{r}}sin(w\bar{r} + \theta_1)cos(\bar{g}) \\ \frac{d}{dt}\bar{g} = Ke^{-c\bar{r}}cos(w\bar{r} + \theta_2)sin(\bar{g}) \end{cases}$$

Define $u(t,x) = V_{\overrightarrow{\xi(t)}} + w(t,x), P_{\xi_i(t)}w(t) = 0, i = 1, 2$. We define the remainder function as

$$w_t - \alpha\partial_{xx} - \beta w - f'(V_{\vec{\xi}})w = \sum_{k=1}^{2} r'_k\varphi_{\xi_k}^r + \sum_{k=1}^{2} g'_k\varphi_{\xi_k}^g + \phi(\overrightarrow{\xi}) + G(\overrightarrow{\xi}, w)$$

where $\phi(\overrightarrow{\xi}) = \phi(\overrightarrow{\xi}, x) := f(V_{\vec{\xi}}(x)) - \sum_{k=1}^2 f(V_{\xi_k}(x))$ and $G(\overrightarrow{\xi}, w) := f(\overrightarrow{\xi} + w) - f(\overrightarrow{\xi}) - f'(\overrightarrow{\xi})w$ is of second order in w: $G(\overrightarrow{\xi}, 0) = G'_w(\overrightarrow{\xi}, 0) = 0$. This is equivalent to

$$w_t - \alpha\partial_{xx} - \beta w - f'(V_{\vec{\xi}})w = h(\overrightarrow{\xi}, w) + \phi(\overrightarrow{\xi}) + G(\overrightarrow{\xi}, w)$$

where

$$h(\overrightarrow{\xi}, w) = -\sum_{k=1}^{2} C^{-1}(\xi, w)[[< \phi(\overrightarrow{\xi}) + G(\overrightarrow{\xi}, w), \psi^r_{\xi_k} > + < \Psi_k(\overrightarrow{\xi})w, \psi^r_{\xi_k} >]\varphi^r_{\xi_k}]$$
$$- \sum_{k=1}^{2} C^{-1}(\xi, w)[[< \phi(\overrightarrow{\xi}) + G(\overrightarrow{\xi}, w), \psi^g_{\xi_k} > + < \Psi_k(\overrightarrow{\xi})w, \psi^g_{\xi_k} >]\varphi^g_{\xi_k}]$$

$$\Psi_k(\overrightarrow{\xi}) := f'(V_{\overrightarrow{\xi}}) - f'(V_{\xi_k})$$

$$C(\overrightarrow{\xi}, w) = \begin{pmatrix} [< \varphi^r_i, \psi^r_j > -\delta_{ij} < w, \partial_x \psi^r_j >]^2_{i,j=1} & [< \varphi^g_i, \psi^r_j >]^2_{i,j=1} \\ [< \varphi^r_i, \psi^g_j >]^2_{i,j=1} & [< \varphi^g_i, \psi^g_j > -\delta_{ij} < w, \partial_x \psi^g_j >]^2_{i,j=1} \end{pmatrix}$$

Finally using the orthogonality conditions we can rewrite the PDE as

$$w_t - (\overrightarrow{\xi})w = h(\overrightarrow{\xi}, w) + \phi(\overrightarrow{\xi}) + G(\overrightarrow{\xi}, w)$$

where $\mathcal{L}(\overrightarrow{\xi}) := \alpha\partial_x^2 - \beta - f'(V_{\overrightarrow{\xi}}) - P_{\overrightarrow{\xi}}$ and $P_{\overrightarrow{\xi}}\theta := \sum_{i=1}^{2} P_{\xi_i}\theta$ which we can solve and plot using chebfun. Below is a phase portrait with a few trajectories.

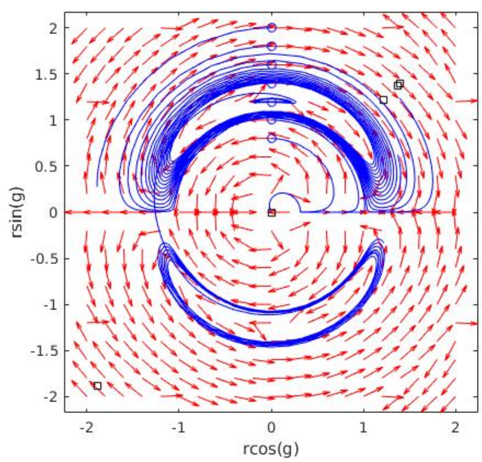

Figure 3: Phase portrait graphed using tspan=[0,1]*700*pi, y0 = [-r;0;r;pi/2] (with r = 1,0.9,0.8,0.7,0.6,0.5,0.4) and relative and absolute tolerance of 1e-5. The unusual trajectory was generated with r = 0.6